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LETTER TO THE EDITOR

Isolated collisionless uniform spherical solution of Boltzmann's equation

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Abstract. A strict solution of Boltzmann's equation without collision term has been found under the initial condition of a uniform spherical group of particles, between which there is only gravitation interaction and the initial velocity of which obeys the Maxwell velocity distribution law. No approximation, except for the collision term, is made in deriving the solution.

The motion of classical particles, such as atoms, molecules and stars, is continuous and can be described by a continuous curve in the phase space; the motion of a group of classical particles can be described by a distributed function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ of the phase space, where $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is the number of particles at time t and in the unit of volume near $\boldsymbol{r}$ and $\boldsymbol{v}$ in phase space. From the continuous equation of the phase space, we can derive the well known Boltzmann equation

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\frac{\partial f}{\partial \boldsymbol{v}}=f_{\mathrm{c}} \tag{1}$$

where the first three terms are known and the last term is the collision term.

If the density of the particle group is small enough, we can make an approximation, neglecting the collision term. This is the only approximation made in this letter. Then, (1) becomes

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\frac{\partial f}{\partial \boldsymbol{v}}=0. \tag{2}$$

However, it is still difficult to find the solution of (2), because $\mathrm{d}v/\mathrm{d}t$ in (2) is acceleration, which reflects the interaction between the particles, and if we take gravitational interaction into consideration, $\mathrm{d}v/\mathrm{d}t$ will include the integration of f, making (2) a non-linear differential-integral equation. Some physicists, such as Pauli (1973) and Lynden-Bell (1967), have discussed the stable solution of (2).

Under the initial condition of a uniform sphere with Maxwell velocity distribution, we found a strict solution of (2).

The explicit formulation of the initial condition is

$$f(v, r, t=0)=M^*(v)N(r, t=0) \tag{3}$$

where $M^*(v)$ is the Maxwell velocity distribution

$$M^*(v)=(m/2kT)^{3/2})^{3/2}\exp(-mv^2/2kT). \tag{4}$$

Here m is the mass of the particle, k the Boltzmann constant and T the absolute temperature. $N(r, t=0)$ in (3) is the density function of the particle in the space at the initial time

$$N(r, t=0) = \begin{cases} n_0 & \text{if } r \leqslant R \\ 0 & \text{if } r > R \end{cases} \tag{5}$$

where n_0 and R are constants.

Let us try to find the solution of (2) in the form:

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = M^*(v) N(r, t=0) \sum_{i=0}^{\infty} (C_t T)^i \sum_{j,l=0}^{[i/2]} a_{ijl} (C_v V \cos\alpha)^{i-2j} (C_r r)^{i-2l} \tag{6}$$

where the constants C_t, C_v and C_r are

$$C_t = (\tfrac{2}{3}\pi m n_0 G)^{1/2} \tag{7}$$

$$C_v = (m/kT)^{1/2} \tag{8}$$

$$C_r = C_t C_v = \left(\frac{2\pi}{3kT} m^2 n_0 G\right)^{1/2} \tag{9}$$

where G is the gravitational constant. For a_{ijl} in (6) the coefficients are yet to be found. j and l are only summed to the integral part of $i/2$.

Note

$$\int M^*(v) v^{2n} \cos^{2n}\alpha \, \mathrm{d}^3 v = (2n-1)!! C_v^{-2n} \tag{10}$$

$$\int M^*(v)(C_v V \cos\alpha)^{2n-1} \, \mathrm{d}^3 v = 0 \tag{11}$$

where $(2n-1)!!$ is defined by

$$(2n-1)!! = (2n-1)(2n-3) \ldots 1$$

and

$$\begin{aligned} &(-1)!! = 1 \\ &m!! = 0 \qquad \text{if } m < -1 \end{aligned} \tag{12}$$

and we can get

$$\begin{aligned} \rho(\boldsymbol{r}, t) &= \int f(\boldsymbol{r}, \boldsymbol{v}, t) \, \mathrm{d}^3 v \\ &= N(r, t=0) \sum_{\xi=0}^{\infty} (C_t T)^{2\xi} \sum_{\eta,\zeta=0}^{\xi} a_{2\xi,\eta,\zeta} (2\xi - 2\eta - 1)!! (C_r R)^{2\xi-2\zeta}. \end{aligned} \tag{13}$$

Now, we assume that ρ has the following form:

$$\rho(\boldsymbol{r}, t) = \begin{cases} \rho(t) \ldots r \leqslant R \\ 0 \ldots r > R \end{cases} \tag{14}$$

exactly, we assume

$$\sum_{\eta=0}^{\xi} a_{2\xi,\eta,\zeta} (2\xi - 2\eta - 1)!! = \delta_{\xi\zeta} \tag{15}$$

then

$$\rho(t) = N(r, t=0) \sum_{\xi=0}^{\infty} (C_t T)^{2\xi}. \tag{16}$$

We know

$$\mathrm{d}\boldsymbol{v}(r,t)/\mathrm{d}t = G\frac{M(r,t)}{r^3}(-\boldsymbol{r}) \tag{17}$$

where

$$M(r,t) = 4\pi\int_0^r \rho(t)r'^2\,\mathrm{d}r'. \tag{18}$$

Thus, we can substitute (6) into (2), and obtain a recurrence forr ula about a_{ijl}:

$$ia_{ijl} + 2\sum_\alpha [(i-2j+1)a_{i-2\alpha-1,j-\alpha-1,1-\alpha} - a_{i-2\alpha-1,j-\alpha,1-\alpha-1}] + (i-2l+1)a_{i-1,j,l-1} = 0. \tag{19}$$

Using (19), we can obtain a series of coefficients a_{ijl} from the initial condition $a_{000}=1$. Then, analysing the relations between these coefficients, we obtain their general formula:

$$a_{ijl} = (-1)^{i+j}\frac{(i-1)!2^{i-j-2l}}{l!j!(i-2l)!(i-l-2j)!}. \tag{20}$$

We define $a_{ijl}=0$ if any $p<0$, where p is $(i-l)$ or l or j or $(i-2l)$ or $(i-l-2j)$ of the right-hand side of (20).

We have already found the solution of (2). What remains is to prove that a_{ijl} in (20) obeys conditions (15) and (19). We only need to prove two mathematical identities to achieve this end.

Identity 1. For any positive integers b and l which do not make any factorial of the following equation equal to zero, we have

$$\sum_{j=0}^{2b-l}\frac{(-1)^j 2^{4b-j-4l}(4b-2j-1)!!}{(b-l)!j!(4b-2l-ij)!} = \delta_{lb}. \tag{21}$$

Qiu Zhaoming at the Institute of Theoretical Physics, Academia Sinica and Professor Sun Yifeng at the Mathematical Department, Jilin University prove this identity using different methods. We only introduce Qui's method here.

Proof.

$$\sum_{j=0}^{2b-l}(-1)^j\frac{2^{4b-j-4l}(4b-2j-i)!!}{(b-l)!j!(4b-2l-2j)!}$$

Let $j=2b-l-i$.

$$\frac{2^{2b-2l}}{(2b-l)!(b-l)!}\sum_{i=0}^{2b-l}\frac{(-1)^{l+i}(2b-l)!}{i!(2b-l-i)!}(i+\tfrac{1}{2})(i+\tfrac{3}{2})\cdots(i+l-\tfrac{1}{2})$$

$$=\frac{2^{2b-2l}}{(b-l)!(2b-l)!}\frac{\mathrm{d}}{\mathrm{d}\chi^l}\sum_{i=0}^{2b-l}(-1)^i C^i_{2b-l}\chi^{-i-1/2}|_{\chi=1}$$

$$=\frac{2^{2b-2l}}{(b-l)!(2b-l)!}\frac{\mathrm{d}^l}{\mathrm{d}\chi^l}[\chi^{-1/2}(1-\chi^{-1})^{2b-l}]_{\chi=1}$$

$$=\frac{2^{2b-2l}}{(b-l)!(2b-1)!}\frac{\mathrm{d}^l}{\mathrm{d}\chi^l}[\chi^{-2b+l-1/2}(\chi-1)^{2b-l}]_{\chi=1}$$

$$=\delta_{bl}.$$

Identity 2. For any positive integers i, j and l which do not make any factorial of the following equation equal to zero, we have

$$\sum_{\alpha=0}^{\min[l,j]} \frac{(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha)!(i-l-2j+\alpha-1)!} + \sum_{\alpha=0}^{\min[l,(j-1)]} \frac{2(i-2j+1)(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha-1)!(i-l-2j+\alpha+1)!} = \frac{(i-l)!}{l!j!(i-l-2j)!}. \tag{22}$$

Professor Sun Yifeng at Jilin University and Professor Wu Zhende at Hebei Normal University have applied the following method to prove identity 2.

Proof. If $l \geqslant j$, then

$$\begin{aligned}
&\sum_{\alpha=0}^{j} \frac{(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha)!(i-l-2j+\alpha-1)!}+2(i-2j+1) \\
&\qquad\times\sum_{\alpha=0}^{j-1} \frac{(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha-1)!(i-l-2j+\alpha+1)!} \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}-\sum_{\alpha=0}^{j-1} \frac{2(i-l-\alpha-2)!(-2)^{\alpha}}{(l-\alpha-1)!(j-\alpha-1)!(i-l-2j+\alpha)!} \\
&\qquad+2(i-2j+1)\sum_{\alpha=0}^{j-1} \frac{(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha-1)!(i-l-2j+\alpha+1)!} \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}+\sum_{\alpha=0}^{j-1} \frac{2(i-l-\alpha-2)!(-2)^{\alpha}}{(l-\alpha-1)!(j-\alpha-1)!(i-l-2j+\alpha)!} \\
&\qquad\times\left(-1+\frac{(i-2j+1)(2-l-\alpha-1)}{(l-\alpha)(i-l-2j+\alpha+1)}\right) \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}+\sum_{\alpha=0}^{j-1} \frac{2(i-l-\alpha-2)!(-2)^{\alpha}}{(l-\alpha-1)!(j-\alpha-1)!(i-l-2j+\alpha)!} \\
&\qquad\times\left(\frac{(i-l-\alpha-1)(i-2j+1)-(i-l-\alpha-1-2j+2\alpha+2)(l-\alpha)}{(l-\alpha)(i-l-2j+\alpha+1)}\right) \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}+\sum_{\alpha=0}^{j-1} \frac{(2i-l-\alpha-2)!(-2)^{\alpha}}{(l-\alpha-1)!(j-\alpha-1)!(i-l-2j+\alpha)!} \\
&\qquad\times\left(\frac{(i-l-\alpha-1)(i-2j-l+\alpha+1)+2(j-\alpha-1)(l-\alpha)}{(l-\alpha)(i-2j-l+\alpha+1)}\right) \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}+\sum_{\alpha=0}^{j-1} \frac{2(i-l-\alpha-1)!(-2)^{\alpha}}{(l-\alpha)!(j-\alpha-1)!(i-l-2j+\alpha)!} \\
&\qquad-\sum_{\alpha=0}^{j-2} \frac{2(i-l-\alpha-2)!(-2)^{\alpha+1}}{(l-\alpha-1)!(j-\alpha-2)!(i-l-2j+\alpha+1)!} \\
&\quad=\frac{(i-l-1)!}{l!j!(i-l-2j-1)!}+\frac{2(i-l-1)!}{l!(j-1)!(i-l-ij)!}
\end{aligned}$$

$$
= \frac{(i-l-1)!}{l!(j-1)!(i-l-2j-1)!}\left(\frac{1}{j}+\frac{2}{i-l-2j}\right)
$$

$$
= \frac{(i-l)!}{l!j!(i-l-2j)!}.
$$

For the case of $1 \leqslant j-1$, the method is similar.

With these identities, it is trivial to prove that (6) and (20) are the solutions of (2).

Boltzmann's equation is one of the most elementary equations in statistical mechanics, especially as it plays a very important part in the statistical mechanics of a non-equilibrium state. However, it cannot be completely solved due to the difficulties in mathematics.

Under a collisionless approximation, this letter reveals how an isolated spherically symmetrical uniform group of particles is evolved by its gravitational force. This is an exact solution of the collisionless Boltzmann's equation under the above-mentioned initial condition. It is our hope that the solution will be favourable for the development of the statistical theory of the non-equilibrium state and its applications. In fact, we have already quoted part of the result of the present letter in another paper (Zhang 1987) in a discussion of the origin of stars and have obtained some satisfactory results.

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